Algebraic Topology for Knowledge Representation in Analogy Solving

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Abstract. We propose a computational model for analogy solving based on a *topological* formalism of representation. The source and the target analogs are represented as *simplexes* and the analogy solving is modeled as a topological *deformation* of these simplexes along a *polygonal chain* and according to some constraints. We apply this framework to the resolution of IQ-tests typically presented as *"given A, B and C, find D such that A is to B what C is to D"*.

1 Introduction

In this paper, we present a topological framework for knowledge representation based on the concept of simplicial complex. We present then the ESQIMO system which is the application of this framework to an analogy solving problem.

The underlying idea developed here is that spatial relationships and more precisely topological relationships such as neighbor, border, dimension, obstruction, deformation, separabitily, path, etc, enable the building and structuration of knowledge representation. More precisely, we explore the possibility of a topological representation to support analogy and we take the elementary spatial entities to be simplicial complexes. The analogy solving between a source and a target domain is then modeled as a topological transformation of the representation of the source into the representation of the target in some underlying abstract space of knowledge representation.

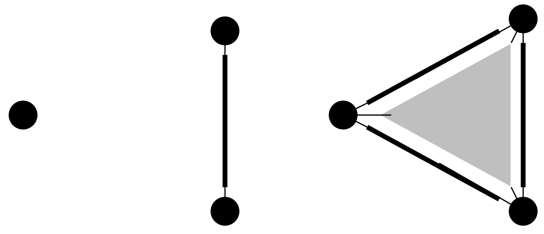
2 Topological Representation of Knowledge

Topology studies objects and properties that are invariant under continuous deformations. Combinatorial topology focuses on the study of a finite sets of objects satisfying some spatial relations, and algebraic topology develops the application of algebraic tools to topological problems. The combinatorial algebraic topological (**CAT**) approach is thus attractive for constructivist models and applications.

2.1 Simplicial Complexes

Simplicial complexes are topological abstract structures that generalize the notion of graph. Indeed, all complexes of dimension less than 2 are graphs. The following definition is standard in algebraic topology.

Definition 1 (Abstract simplicial complex)
An abstract simplicial complex is a couple (V, K) where V is a set of elements called vertices of the complex and K is a set of finite parts of V such that if $s \in K$, then all the parts $s' \subseteq s$ belongs also to K. The elements of K are called abstract simplexes. The dimension of a simplex s is equal to $Card(s) - 1$. The dimension of the complex is the dimension of its biggest simplex [13, 15].



(a) 0-simplex (b) 1-simplex (c) 2-simplex

Figure 1. Geometrical representation of p-simplexes for p varying from 0 to 2.

A p-simplex s is noted: $s = \langle v^0 v^1 ... v^p \rangle$, where $v^i \in V$, the figure 1 shows the geometrical representation of 0, 1 and 2-simplexes. A complex is a set of sets closed for the inclusion and the intersection. Thus, simplicial complexes are particularly attractive to generalize semantic networks by keeping the possibility to express hierarchies like in a relational graph (a hierarchical structure is highly recommended and trees are often not sufficient for that [14]).

2.2 Knowledge Representation with Simplicial Complexes.

2.2.1 Representation of a Binary Relation

Atkin already proposed to represent a binary relation λ between two sets with a simplicial complex: it is the **Q-Analysis** [2, 3, 19]. Q-Analysis have been used to model traffics [20], interactions between agents [5, 22, 4, chap. 8], position analysis at chess [1] and social relations [2, 11, 4].

Let Λ be the incidence matrix of a binary relation $\lambda \subset A \times B$. Let $a \in A$, and the set B_a of $b_i \in B$ such that $(a, b_i) \in \lambda$. The set B_a can be directly read from Λ, as the a-column (see table 1).

We represent the elements b_i of B_a as vertices and a as a simplex build on these vertices. The dimension of the simplex S_a representing a depends on the number of vertices in B_a.

The whole matrix Λ can then be represented as a simplicial complex containing all the simplexes representing each element $a_i \in A$, we note it $K_A(B, \lambda)$ (see figure 2.2.1).

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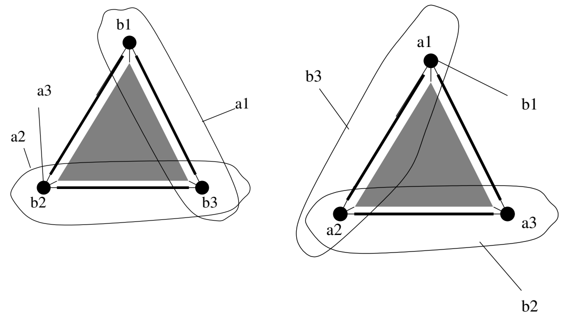
λ	a_1	a_2	a_3
b_1	1	0	0
b_2	0	1	1
b_3	1	1	0

Table 1. Incidence matrix associated with λ.



(a) Simplicial representation of λ taking b_i as vertices and a_i as simplexes

(b) Dual simplicial representation of λ taking a_i as vertices and b_i as simplexes

Figure 2. Simplicial representation of a binary relation λ. We have $\lambda(a_1) = \{b_1, b_2\}$. So we represent a_1 as a 1-simplex, b_1 and b_2 being its two vertices.

Likewise, we can represent Λ^{-1} with the dual simplicial complex $K_B(A, \lambda^{-1})$. In this case, the elements a_i are taken as vertices and the elements b_i are represented as simplexes (see figure 2.2.1). We say that $K_A(B, \lambda)$ and $K_B(A, \lambda^{-1})$ are conjugates, they contain the same information but present it in a different and complementary way.

We say that two simplexes σ_1 and σ_2 are q-connected if there is a polygonal chain of dimension q that connects σ_1 with σ_2.

Definition 2 (Polygonal chain)
Let $\alpha = (\sigma_0, \sigma_1, ..., \sigma_n)$ be a sequence of simplexes belonging to a complex K. It is called a polygonal chain of origin σ_0 and end σ_n if for all couple (σ_i, σ_{i+1}), $\sigma_i \cap \sigma_{i+1} \neq \emptyset$. The dimension of α is the smallest dimension of $\sigma_i \cap \sigma_{i+1}$.

Any p-simplex is p-connected to himself with a 0-chain.

2.2.2 *Representation of a Set of Predicates*

We extend the Q-Analysis to allow the representation of sets of predicates as a simplicial complex too. The idea, which is very simple, is to take a set of predicates $P = \{p_1, p_2, ..., p_n\}$ and represent the binary relation $\lambda \subset A \times P$ such that $(a_i, p_j) \in \lambda$ if $p_j(a_i)$ holds.

Take for example the set of integers $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the set of predicates $P = \{p_1, p_2, p_3, p_4\} = \{$parity, oddity, primality, multiple of 3$\}$. The incidence matrix of λ is then obviously the one given on table 2. We can represent the dual complex of λ, each element $a_i \in A$ being a simplex build with vertices $p_i \in P$. this dual representation enlighten the fact that elements $4, 8, 10$ have exactly the same representation when taking these few predicates. A representation

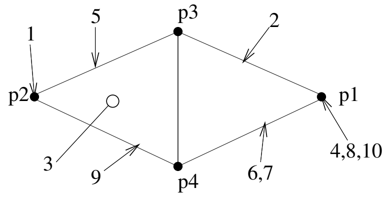


Figure 3. Dual complex associated with $\lambda \subset \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \times \{p_1, p_2, p_3, p_4\}$ where we can see that the integers $4, 8$ and 10 are identical with respect to these criteria.

	p_1	p_2	p_3	p_4
1	0	1	0	0
2	1	0	1	0
3	0	1	1	1
4	1	0	0	0
5	0	1	1	0
6	1	0	0	1
7	0	1	1	0
8	1	0	0	0
9	0	1	0	1
10	1	0	0	0

Table 2. Incidence matrix associated with λ in the numbers example.

based upon simplicial complexes associates the same simplex to elements of A that cannot be distinguished. In other words, two elements will be separated only if there is at least one predicate that allows the differentiation. The same situation occurs with the dual complex.

Two simplexes that have a smaller k-simplex in common are said to share a k-face. In terms of representation, it means that *they have k features in common*. As Freska emphasized it, we call here for the use of discriminating features rather than for precise characterization in terms of universally applicable reference system [8].

We can say that the identity of an element is represented by the features he shares with others and also by the ones that are specific to it [18].

3 The ESQIMO System for Analogy Solving

3.1 Solving an Analogy

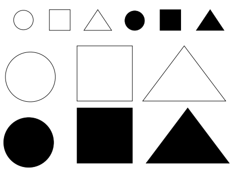
To model a process of analogy solving on the basis of the previous topological setting, we chose a small and paradigmatic application domain [25]. The task is to answer a typical IQ-test by giving an element called D such that it completes a four-term analogy with three other given elements A, B and C: "find D such that it is to C what B is to A". This kind of analogy solving has already been studied by Evans [6], but in our work the solution has to be build from scratch since no set of possible solutions is given to choice. We call this kind of IQ-test-like problems, *non supervised*. This four-term analogy solving is usually decomposed into four steps [6]:

- Find the possible relations R_{AB} between A and B.
- Find the possible relations R_{AC} between A and C.
- Apply R_{AB} to C only on a domain determined with R_{AC}.
- Verify the symmetry by applying R_{AC} to B.

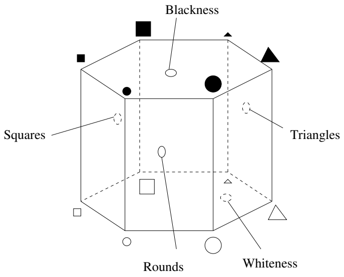
To solve a four-term analogy, we propose to represent each figure by a simplex and the relation between the first two figures by a path (a polygonal chain) into the problem space (a complex). Buildong the fourth figure from the third will thus be deforming this third figure according to the precedent path.

3.2 The Objects of ESQIMO

Usually, IQ-tests are given in terms of geometrical elements so that they can express many different properties at the same level and still stay simples. We chose a geometrical universe similar to the one investigated in [26] of twelve basic elements $E = \{e1, ..., e_{12}\}$, as shown on figure 4. These elements are all the possible combinations of the seven properties (or predicates): $P = \{p_1, ..., p_7\} = \{\text{round, square, triangle, white, dark, big, small}\}$.

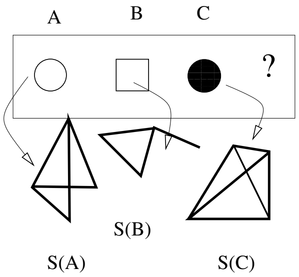


(a) Elements of the universe Ω of ESQIMO, respectively called e_1 to e_{12} starting from the top left element

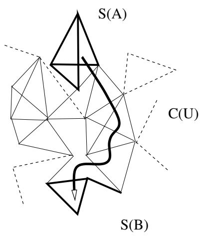


(b) A 2D view of the dual complex $K'(Omega)$, the elements of E are the vertices and the properties $p_i \in P$ are simplexes of $K'(\Omega)$. Notice that the 6-simplex representing the property of blackness is normally 5-dimensional

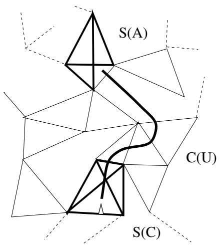
Figure 4. Elements manipulated by ESQIMO and their representation as a simplicial complex.



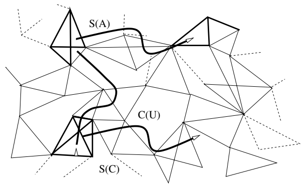
(a) Figures A, B and C are represented as simplexes belonging to $K'(\Omega)$



(b) Transformation T_{AB} is modeled as a polygonal chain from S_A to S_B into $K'(\Omega)$



(c) The domain of S_C to which apply T_{AB} is determined with the help of T_{AC}



(d) S_D is the deformation of S_C along T_{AB} applied to the relevant domain of S_C

Figure 5. Four steps of ESQIMO's algorithm to solve IQ tests in the case of simple figures A, B and C respectively represented as the simplexes S_A, S_B and S_C.

These two sets are the only knowledge used by ESQIMO to solve the tests. We can represent this knowledge with a simplicial complex $K(\Omega)$ or its conjugate $K'(\Omega)$ (see figure 4) by representing the binary relation $\lambda \subset A \times P$ such that $(a_i, p_j) \in \lambda$ if $p_j(a_i)$ holds. The complex $K'(\Omega)$ is then the space of the problem in which ESQIMO solves analogies by deforming simplexes into others.

3.3 Algorithm

3.3.1 Representing the Problem

When a problem is presented, each figure A, B and C is composed of one or more elements $e_i \in E$. Each element e_i can be represented as a simplex of $K(\Omega)$, the properties p_j such that $p_j(e_i)$ holds, being its vertices. Thus, a simple figure (composed of only one element) will be represented as a simplex and a composed figure (more than one element) will be represented with a set of simplexes. The problem is now to find a relation between the (set of) simplex(es) representing A and the (set of) simplex(es) representing B and apply it to the (set of) simplex(es) representing C. Note that the representations of A, B and C are all included into the complex $K'(\Omega)$.

3.3.2 Case of simple figures

In the case of simple figures, the transformation T_{AB} is seen as a polygonal chain from S_A to S_B in $K(\Omega)$. An elementary step linking S_i to S_{i+1} in a chain is then viewed as an elementary transformation $T_{S_i,S_{i+1}}$. A polygonal chain from S_A to S_B is then a transformation of A into B given by: $T_{S_l,S_B} \circ ... \circ T_{S_A,S_1}$.

If there are several chains, then we say that there are several possible relations between A and B. We can choose to minimize the number of possible solutions, by giving a higher priority to polygonal chains that are short and of higher dimension, that corresponds to choose a transformation that requires less steps and that preserves more properties. This is comparable with selecting a *best* solution according to some measure of satisfaction like in [6].

To apply T_{AB} to S_C we have to extend the domain of T_{AB}, and so extend T_{AB} to T'_{AB} such that $T'_{AB}(S_C) = S_D$ and $T'_{AB}(S_A) = S_B$ (close to a simplicial application [13, 15]). There are different possible strategies to determine the domain of $S(C)$ on which we can apply T_{AB}. Several strategies have been implemented considering only the things that changed between $S(A)$ and $S(C)$, or considering only the invariants between them, or some other hybrid methods.

3.3.3 Case of composed figures

For composed figures, the transformations can be of several types: destruction, creation, metamorphosis, division, junction (like in the changes introduced by Hornsby [18]). We first pair the simplexes of $\{S_A\}$ with those of $\{S_B\}$ and look for transformations between the simplexes of each pair. The transformation T_{AB} is then the parallel application of the transformation found for each pair.

There are many possible pairings leading to different or to the same solution. The only constraint we need is that all the vertices and faces of $S(B)$ are paired with vertices from $S(A)$. A pairing of a vertex of $S(A)$ with $\emptyset$ means its destruction, the pairing of a vertex of $S(A)$ with one vertex of $S(B)$ means its transformation and the pairing of a vertex from $S(A)$ with several vertices of $S(B)$ means its duplication with transformation.

3.4 Examples of Analogy Solving with ESQIMO

We give three examples of IQ-test solving with ESQIMO on figures 6, 7 and 8. In the first example, we ask ESQIMO to solve the IQ-test with the call of the function `Resolve` with the pairing parameters `App2` and `AppApp2` as shown below (for more details see [24]). The three given figures A, B and C are defined in terms of e_i elements of E. As seen on figure 6, A is composed of a white small circle plus a white small square.

```
A={e1,e2};
B={e7,e5};
C={e3,e1};
Resolve[A,B,C,App2,AppApp2]
```

Here, A is a composed figure, its representation corresponds to the set of simplexes $S_A = \{\langle p_1, p_4, p_7\rangle, \langle p_2 p_4 p_7\rangle\} = \{\langle S_A^1, S_A^2\rangle\}$. Likewise, the representations of B and C are respectively, $S_B = \{\langle p_1, p_4, p_6\rangle, \langle p_2 p_5 p_7\rangle\} = \{\langle S_B^1, S_B^2\rangle\}$ and $S_C = \{\langle p_3, p_4, p_7\rangle, \langle p_1 p_4 p_7\rangle\} = \{\langle S_C^1, S_C^2\rangle\}$.

`App2` is a strategy for the pairing between the set of simplexes of A, and the set of simplexes of B that gives the following pairing:

$$(S_A^1 \to S_B^1), (S_A^2 \to S_B^2)$$

ESQIMO gives output about intermediate results such as pairings, the result of applying strategy `App2` is given by the following output:

```
Choose[AssocSet[FromTo[1,{1}],FromTo[2,{2}]],
       AssocSet[FromTo[1,{1}],FromTo[2,{2}]]]}
```

Where an `AssocSet` is a set of pairings and `FromTo` is a pairing, which means also an elementary transformation `From` the first element of the pair `To` the second one. For each paring, an elementary transformation is proposed, depending on the heuristic used which is another parameter (that is internally settled untill now [24]). we call them respectively T_1 and T_2. Then, the pairing strategy `AppApp2` is used to apply these elementary transformations to the elements of the set of simplexes representing C, it proposes to apply in parallel:

$$T_1(S_C^1)//T_2(S_C^2)$$

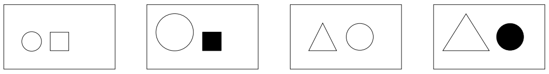
The corresponding output is:

```
Par[Domain[1,Seq["D-elem"[SmallQ->0,BigQ->1]]
    ,{e3}],
    Domain[2,Seq["D-elem"[WhiteQ->0,BlackQ->1]]
           ,{e1}]]}
```

where `Par` means a parallel application and `Seq` a sequential application of the elementary transformation described in terms of change of properties (or predicates). Finally the solution is composed of two elements represented by the simplexes $S_D = \{\langle p_3, p_4, p_6\rangle, \langle p_1, p_5, p_7\rangle\} = \{e_9, e_4\}$ (see figure 6), the corresponding output is:

```
Choose[{e9,e4}]
```

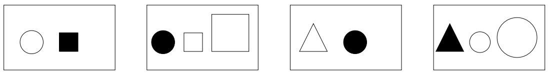
All along the solving process, ESQIMO uses the prefix `Choose` in all its outputs. That is because many different solutions are possible and acceptable for a psychological plausibility. ESQIMO can compute many solutions in parallel without selecting a *best* one, in that case there are many solutions that the user can `Choose` at the end.



(a) (b) (c) (d)

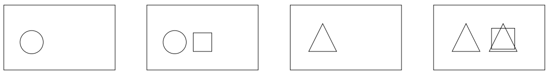
Figure 6. The first element becomes bigger and the second becomes black.

The two other examples are solved with the same pairing strategies and are not detailed here.



(a) (b) (c) (d)

Figure 7. The first element becomes black and the second becomes white, is duplicated and one of the duplicates is bigger.



(a) (b) (c) (d)

Figure 8. The first element is duplicated and one duplicate is squared. When squared, the property of triangleness is not taken off, this creates then an unstable solution, called a monster.

3.5 Discussion and Conclusion

Many choices made in ESQIMO's algorithm can be discussed. In fact, they can be seen as additional strategies parameterizing the ESQIMO kernel. For example:

- The description of the properties of each figure in terms of predicates can be a problem for properties such as position. We could give each possible position a predicate that could be true or false.

- The way we associate a transformation to a given polygonal chain is not unique. In particular, our transformations could be called $0-$degree since they preserve the minimum of topological properties along a chain. The next step consists in pairing higher-order structures between the sets of simplexes.
- The way we determine the domain of S_C on which to apply T_{AB} can also lead to different strategies depending on whether we consider only the intersection between S_A and S_C or the whole S_C.
- The measure of satisfaction to select a *best* solution is here to take the shorter and wider polygonal chain between the two complexes. Other measures of satisfaction can be tested.

Furthermore, note that our formalization of IQ-test problems does not depend on their geometrical nature. Indeed, only the representational level is based on topology while the objects manipulated by the system could have been non geometrical. We could, for example try ESQIMO on verbal IQ-tests more like in the Copycat system [16].

Different computational models have been developed to model analogy solving. Among them, the ANALOGY system proposed by Evans [6, 12], the SME system proposed by Falkenhainer to illustrate Gentner's theory for analogy [7, 9], the ARCS system developed by Thagard and Holyoak to simultaneously satisfy the structural, semantic and pragmatic constraints. We can hardly compare these systems to ESQIMO in terms of performances since we only studied intra-domain analogies with the only structural constraint in this first work. Our contribution lies principally in the search for a new representational structure to model analogy, which has often been described in terms of a morphism. The topological structure of representation can be seen as a hybrid structure between a purely symbolic and a purely analogical approach.

ESQIMO has been prototyped in the `Mathematica` [27] programming language and we find the results presented here already surprisingly satisfying with respect to the simplicity of the underlying machinery. This clearly motivates further investigations and a more complete version is being implemented in the `ML` programming language [21]. Indeed, we intend to explore a possible use of the notions of homotopy and cobordism to formalize the concept of similarity between polygonal chains or between paths on topological representations. This could lead to a generalization of our topological model for analogy.

Finally, the representational formalism presented here has been considered in the wider field of diagrammatic reasoning [10]. Thus, ESQIMO could also lead to the conception of a toolkit for the assistance to diagrammatic tasks such as system architecture design (software or hardware). More details on the application of our model to diagrammatic reasoning are given in [23], where the construction of our topological representational structure is inspired by Holland's quasi-homomorphism model [17].

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